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ON RUNGE–KUTTA METHODS OF ORDER 10*

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Abstract. An explicit s-stage Runge–Kutta method of order 10 is determined by $s(s+1)/2$ parameters that must satisfy a non-linear algebraic system of 1205 equations. In the literature, solutions for the cases $s = 18$ [A. R. Curtis, J. Inst. Math. Appl., 16 (1975), pp. 35–55] and $s = 17$ [E. Hairer, J. Inst. Math. Appl., 21 (1978), pp. 47–59] were analytically derived, while that for $s = 16$ [D. K. Zhang, Numer. Algorithms, 96 (2024), pp. 1243–1267] was found by numerical search. In the present paper, a family of methods with $s = 15$ is derived.

Key words. minimal number of stages, explicit Runge–Kutta methods

AMS subject classifications. 65L05, 65L06

Runge–Kutta methods (see, e.g., [7, Section 23 and Chapter 3], [16, Chapter. II], [1, Chapter 4], [18, Chapter 3]) have been widely and successfully used to numerically solve ordinary differential equations for over a century [9]. Being applied to a system $\mathrm{d}\boldsymbol{x}/\mathrm{d}t = \boldsymbol{f}(t, \boldsymbol{x})$, in order to forward propagate by a step size h and update the position, $\boldsymbol{x}(t) \mapsto \tilde{\boldsymbol{x}}(t+h)$, where $\tilde{\boldsymbol{x}}(t+h)$ is a numerical approximation to the exact solution $\boldsymbol{x}(t+h)$, an s-stage Runge–Kutta method (characterized by the coefficients a_{ij}, weights b_j, and nodes c_i) would proceed as follows: At first the following system of equations for $\boldsymbol{X}_1, \boldsymbol{X}_2, \ldots, \boldsymbol{X}_s$ is formed,

$$\boldsymbol{X}_i = \boldsymbol{x}(t) + h\sum_{j=1}^{s} a_{ij}\boldsymbol{F}_j, \qquad \boldsymbol{F}_i = \boldsymbol{f}\big(t + c_i h, \boldsymbol{X}_i\big), \qquad i = 1, 2, \ldots, s,$$

which is then solved, followed by the computation of $\tilde{\boldsymbol{x}}(t+h) = \boldsymbol{x}(t) + h\sum_{j=1}^{s} b_j\boldsymbol{F}_j$. In the limit $h \to 0$, all the vectors $\boldsymbol{F}_i$, $1 \le i \le s$, are the same, so it is natural and will be assumed that $\sum_{j=1}^{s} a_{ij} = c_i$ for all i.

A method is said to be of *order* (at least) p if, for sufficiently smooth right-hand side functions $\boldsymbol{f}$, the local truncation error behavior is $\|\boldsymbol{x}(t+h) - \tilde{\boldsymbol{x}}(t+h)\| = \mathcal{O}(h^{p+1})$ as $h \to 0$. It is often desirable to use a method of higher order, as this makes it possible to obtain a solution with a certain level of accuracy with a smaller number of steps. For an s-stage Runge–Kutta method, the maximal possible order is $p = 2s$, achieved by Gauss–Legendre methods [2].

A Runge–Kutta method is said to be *explicit* if $a_{ij} = 0$ whenever $j \ge i$. Then $c_1 = 0$, $\boldsymbol{X}_1 = \boldsymbol{x}(t)$, $\boldsymbol{F}_1 = \boldsymbol{f}\big(t, \boldsymbol{x}(t)\big)$, $a_{21} = c_2$, and $\boldsymbol{X}_2, \boldsymbol{F}_2, \boldsymbol{X}_3, \boldsymbol{F}_3, \ldots, \boldsymbol{X}_s, \boldsymbol{F}_s$ can be computed sequentially by a direct computation. For instance, at the moment of finding $\boldsymbol{X}_3$, the vectors $\boldsymbol{X}_1, \boldsymbol{F}_1, \boldsymbol{X}_2, \boldsymbol{F}_2$ will have already been computed, and

$$\boldsymbol{X}_3 \;=\; \boldsymbol{x}(t) + ha_{31}\overbrace{\boldsymbol{f}\big(t, \underbrace{\boldsymbol{x}(t)}_{\boldsymbol{X}_1}\big)}^{\boldsymbol{F}_1} + ha_{32}\overbrace{\boldsymbol{f}\Big(t + c_2 h, \underbrace{\boldsymbol{x}(t) + hc_2\boldsymbol{f}\big(t, \boldsymbol{x}(t)\big)}_{\boldsymbol{X}_2}\Big)}^{\boldsymbol{F}_2}.$$

Determining the minimal number of stages $s_{\min}(p)$ for which there exists an explicit Runge–Kutta method of order p is a complicated problem, which is currently solved for $p \le 8$: It holds

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that $s_{\min}(\langle 1,2,3,4,5,6,7,8\rangle) = \langle 1,2,3,4,6,7,9,11\rangle$, with the lower bound $s_{\min}(p) \geq p+3$ for $p \geq 8$ [6].

There are known explicit methods of order 10 with 18 stages [12]. For the results with 17 stages, see [14] and, following its structure, [22], as well as [13] with performance traded off for the presence of an embedded method of order 8. For a method with 16 stages, see [27] (although there is not yet a rigorous proof that the method is indeed of order 10, the numerical evidence is overwhelming).

The aim of this work is to construct an explicit 15-stage Runge–Kutta method of order 10. The order conditions are stated in Section 1. Order conditions of two types, Q- and D-types, are considered in Section 2, while in Sections 3 and 4 these are compared and contrasted. A 7-dimensional family of explicit 15-stage Runge–Kutta methods of order 10 is derived in Section 5. Some previously known methods of order 10 are compared to the proposed new one in Section 6.

1. Order conditions. The elementwise product of tensors $\boldsymbol{x}$ and $\boldsymbol{y}$ of the same size will be denoted as $\boldsymbol{x}.\boldsymbol{y}$, e.g., in case of vectors, $(\boldsymbol{x}.\boldsymbol{y})_i = x_i y_i$. The elementwise product of n copies of a column vector $\boldsymbol{x}$ will be written as $\boldsymbol{x}^n$. Let $\mathbf{1}$ be the s-dimensional column vector with all components being equal to 1, let $\boldsymbol{A} = \left[a_{ij}\right]$ be the $s \times s$ matrix with a_{ij} as its matrix element in the ith row and jth column, let $\boldsymbol{b} = \left[b_j\right]$ be the weights row vector, and $\boldsymbol{c} = \left[c_i\right]$ be the nodes column vector.

Given rooted trees $\mathrm{t}_1, \mathrm{t}_2, \ldots, \mathrm{t}_n$, a new tree $[\mathrm{t}_1\mathrm{t}_2\ldots\mathrm{t}_n]$ is obtained by connecting, with n edges, their roots to a new vertex; the latter becomes a new root [7, Section 301], [16, p. 152], [8, p. 44], [15, p. 53]. Consider a vector function $\boldsymbol{\Phi} : \mathrm{T} \to \mathbb{R}^s$ on the set of rooted trees that is recursively defined as $\boldsymbol{\Phi}(\bullet) = \mathbf{1}$ and $\boldsymbol{\Phi}\big([\mathrm{t}_1\mathrm{t}_2\ldots\mathrm{t}_n]\big) = \prod_{m=1}^{n} \boldsymbol{A}\boldsymbol{\Phi}(\mathrm{t}_m)$, where the product of vectors is taken elementwise. This function coincides with the *derivative weights* in [7, Def. 312A], [16, pp. 148 and 151], and it is closely related to the *internal* or *stage weights* $\boldsymbol{A}\boldsymbol{\Phi}(\mathrm{t})$ [8, p. 125] and the *elementary weights* $\boldsymbol{b}\boldsymbol{\Phi}(\mathrm{t})$ [15, p. 55].

A Runge–Kutta method $(\boldsymbol{A}, \boldsymbol{b}, \boldsymbol{c})$ is of order (at least) p if and only if for any rooted tree t, with $|\mathrm{t}| \leq p$, one has [7, Section 315], [16, p. 153], [8, p. 127], [15, p. 56]

$$\boldsymbol{b}\boldsymbol{\Phi}(\mathrm{t}) = 1/\mathrm{t}!\,.$$

Here $|\mathrm{t}|$ is the order of the tree t, i.e., the number of vertices in t. The factorial t! is recursively defined as $\bullet! = 1$ and if $\mathrm{t} = [\mathrm{t}_1\mathrm{t}_2\ldots\mathrm{t}_n]$, then $\mathrm{t}! = |\mathrm{t}| \prod_{m=1}^{n} (\mathrm{t}_m)!$.

2. Q- and D-type order conditions. For any rooted tree t, let $\boldsymbol{Q}(\mathrm{t}) = \boldsymbol{A}\boldsymbol{\Phi}(\mathrm{t}) - \boldsymbol{c}^{|\mathrm{t}|}/\mathrm{t}!$. For a rooted tree $\mathrm{t} = [\bullet^n]$, the vector $\boldsymbol{Q}\big([\bullet^n]\big) = \boldsymbol{q}_n = \boldsymbol{A}\boldsymbol{c}^n - \boldsymbol{c}^{n+1}/(n+1)$ is a standard subquadrature vector [26, p. 1124], [25, p. 558]. It is convenient to define $[\bullet^0] = [\varnothing] = \bullet$. The assumed condition $\boldsymbol{A}\mathbf{1} = \boldsymbol{c}$ coincides with $\boldsymbol{Q}(\bullet) = \boldsymbol{q}_0 = \mathbf{0}$.

Consider a vector space of all finite linear combinations of trees $\mathcal{T} = \oplus_{\mathrm{t}\in\mathrm{T}}\mathbb{R}$. Such a linear combination can be written as $\mathsf{T} = \sum_{\mathrm{t}\in\mathrm{T}} a(\mathrm{t})\mathrm{t}$, with the weights $a(\mathrm{t})$ being non-zero only for finitely many trees t. Let $\mathcal{Q} : \mathrm{T} \to \mathcal{T}$ be a mapping defined as $\mathcal{Q}(\mathrm{t}) = [\mathrm{t}] - [\bullet^{|\mathrm{t}|}]/\mathrm{t}!$. One has $\mathcal{Q}(\bullet) = 0$. All three mappings $\boldsymbol{\Phi}$, $\boldsymbol{Q}$, and $\mathcal{Q}$ can be extended by linearity to linear combinations of trees: $\langle \boldsymbol{\Phi}, \boldsymbol{Q}, \mathcal{Q}\rangle\big(\sum_{\mathrm{t}\in\mathrm{T}} a(\mathrm{t})\mathrm{t}\big) = \sum_{\mathrm{t}\in\mathrm{T}} a(\mathrm{t})\langle \boldsymbol{\Phi}, \boldsymbol{Q}, \mathcal{Q}\rangle(\mathrm{t})$. One has $\boldsymbol{Q}(\mathsf{T}) = \boldsymbol{\Phi}\big(\mathcal{Q}(\mathsf{T})\big)$ for any $\mathsf{T} \in \mathcal{T}$.

The order conditions $\boldsymbol{b}\boldsymbol{\Phi}(\mathrm{t}) = 1/\mathrm{t}!$, whenever $|\mathrm{t}| \leq p$, can be rewritten as quadrature conditions $\boldsymbol{b}\boldsymbol{c}^n = 1/(n+1)$, for $0 \leq n < p$, and order conditions of *Q-type*

$$\boldsymbol{b}(\boldsymbol{Q}(\mathrm{t}_1).\boldsymbol{Q}(\mathrm{t}_2).\cdots.\boldsymbol{Q}(\mathrm{t}_k).\boldsymbol{c}^n) = 0 \quad \text{for } k \geq 1 \text{ and } |\mathrm{t}_1| + |\mathrm{t}_2| + \ldots + |\mathrm{t}_k| + n < p.$$

For any rooted tree t, let a row vector $\boldsymbol{D}(\mathrm{t})$ be defined as

$$D_j(\mathrm{t}) = \big(\boldsymbol{b}.\boldsymbol{\Phi}^{\mathrm{T}}(\mathrm{t})\big)\boldsymbol{a}_{*j} - b_j(1 - c_j^{|\mathrm{t}|})/\mathrm{t}! \quad \text{or} \quad \boldsymbol{D}(\mathrm{t}) = \big(\boldsymbol{b}.\boldsymbol{\Phi}^{\mathrm{T}}(\mathrm{t})\big)\boldsymbol{A} - \big(\boldsymbol{b}.(\mathbf{1} - \boldsymbol{c}^{|\mathrm{t}|})^{\mathrm{T}}\big)/\mathrm{t}!.$$

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For a rooted tree $\mathrm{t} = [\bullet^n]$, the row vector is

$$\boldsymbol{D}\big([\bullet^n]\big) = \boldsymbol{d}_n = (\boldsymbol{b}.\boldsymbol{c}^{n\mathrm{T}})\boldsymbol{A} - \boldsymbol{b}.(\boldsymbol{1} - \boldsymbol{c}^{n+1})^{\mathrm{T}}/(n+1).$$

The properties $B(n)$, $C(n)$, and $D(n)$ (for their definition and also for simplifying assumptions see, e.g., [2, p. 52], [7, Section 321], [16, pp. 175, 182, and 208]) can be formulated in terms of the vectors $\boldsymbol{q}_k$ and $\boldsymbol{d}_k$:

$$B(n): \qquad \boldsymbol{b}\boldsymbol{c}^k = 1/(k+1) \qquad \text{for all } 0 \le k < n, \tag{2.1}$$

$$C(n): \qquad \boldsymbol{q}_0 = \boldsymbol{q}_1 = \ldots = \boldsymbol{q}_{n-1} = \boldsymbol{0}, \tag{2.2}$$

$$D(n): \qquad \boldsymbol{d}_0 = \boldsymbol{d}_1 = \ldots = \boldsymbol{d}_{n-1} = \boldsymbol{0}. \tag{2.3}$$

Given rooted trees $\mathrm{t}_1, \mathrm{t}_2, \ldots, \mathrm{t}_n$, a new tree $\mathrm{t}_1 \cdot \mathrm{t}_2 \cdot \ldots \cdot \mathrm{t}_n$ is obtained by merging their n roots into one vertex, where the latter becomes a new root. One has

$$\boldsymbol{\Phi}(\mathrm{t}_1 \cdot \mathrm{t}_2 \cdot \ldots \cdot \mathrm{t}_n) = \boldsymbol{\Phi}(\mathrm{t}_1).\boldsymbol{\Phi}(\mathrm{t}_2).\cdots.\boldsymbol{\Phi}(\mathrm{t}_n).$$

Let $\mathcal{D} : \mathrm{T}^2 \to \mathcal{T}$ be a mapping defined as $\mathcal{D}(\mathrm{t}, \mathrm{t}') = \mathrm{t} * \mathrm{t}' + [\bullet^{|\mathrm{t}|}] \cdot \mathrm{t}'/\mathrm{t}! - \mathrm{t}'/\mathrm{t}!$ for any rooted trees t and t$'$, where $\mathrm{t}_1 * \mathrm{t}_2 = \mathrm{t}_1 \cdot [\mathrm{t}_2]$ is the beta-product of trees [5], [8, p. 45]. The mapping $\mathcal{D}$ can be viewed as a collection of mappings $\mathcal{D}(\mathrm{t}, \cdot) : \mathrm{T} \to \mathcal{T}$, with $\mathrm{t}' \mapsto \mathcal{D}(\mathrm{t}, \mathrm{t}')$, indexed by a tree t, which is a certain way of formalizing the concept of stumps [8, Section 2.7].

The mapping $\mathcal{D}$ in its second argument can be extended by linearity to linear combinations of trees:

$$\mathcal{D}\Big(\mathrm{t}, \sum_{\mathrm{t}' \in \mathrm{T}} a(\mathrm{t}')\mathrm{t}'\Big) = \sum_{\mathrm{t}' \in \mathrm{T}} a(\mathrm{t}')\mathcal{D}(\mathrm{t}, \mathrm{t}').$$

One has $\boldsymbol{D}(\mathrm{t})\boldsymbol{\Phi}(\mathsf{T}) = \boldsymbol{b}\boldsymbol{\Phi}\big(\mathcal{D}(\mathrm{t}, \mathsf{T})\big)$ for any $\mathrm{t} \in \mathrm{T}$ and $\mathsf{T} \in \mathcal{T}$. The diagram on the right is commutative.

$$\begin{tikzcd}
 & & \mathcal{T} \arrow[dl, "\boldsymbol{Q}"'] \arrow[dr, "\mathcal{Q}"] & \\
\mathbb{R} & \mathbb{R}^s \arrow[l, "\boldsymbol{b}"'] & & \mathcal{T} \arrow[ll, "\boldsymbol{\Phi}"'] \\
 & \mathbb{R}^s \arrow[ul, "\boldsymbol{D}(\mathrm{t})"] & & \mathcal{T} \arrow[ll, "\boldsymbol{\Phi}"'] \arrow[ul, "\mathcal{D}(\mathrm{t},\cdot)"']
\end{tikzcd}$$

The order conditions $\boldsymbol{b}\boldsymbol{\Phi}(\mathrm{t}) = 1/\mathrm{t}!$, whenever $|\mathrm{t}| \le p$, can be rewritten as quadrature conditions $\boldsymbol{b}\boldsymbol{c}^n = 1/(n+1)$, for $0 \le n < p$, and order conditions of *D-type*

$$\boldsymbol{D}(\mathrm{t})\boldsymbol{\Phi}(\mathrm{t}') = 0 \qquad \text{for all rooted trees t and t}' \text{ such that } |\mathrm{t}| + |\mathrm{t}'| \le p.$$

In the order conditions of Q-type, there could be several $\boldsymbol{Q}(\mathrm{t})$ vectors, while a condition of D-type has only one $\boldsymbol{D}(\mathrm{t})$ since a rooted tree can have many branches but only one root.

3. The dual of a method. This concept of the dual of a method is defined as follows:

DEFINITION 3.1. *Consider an s-stage Runge–Kutta method $\mathcal{M} = (\boldsymbol{A}, \boldsymbol{b}, \boldsymbol{c})$ with all the weights being non-zero, i.e., $b_j \neq 0$, for all $1 \le j \le s$, and which also satisfies $D(1)$, i.e., $\boldsymbol{b}\boldsymbol{A} = \boldsymbol{b}.(\boldsymbol{1} - \boldsymbol{c})^T$.*

A method dual *to $\mathcal{M}$ is the s-stage method $\mathcal{M}^* = (\boldsymbol{A}^*, \boldsymbol{b}^*, \boldsymbol{c}^*)$, where*

$$c_i^* = 1 - c_{s+1-i}, \quad b_j^* = b_{s+1-j}, \quad \textit{and} \quad a_{ij}^* = b_{s+1-j}a_{s+1-j,s+1-i}/b_{s+1-i},$$

for all $1 \le i, j \le s$. A method $\mathcal{M}$ is called self-dual *if $\mathcal{M}^* = \mathcal{M}$.*

STATEMENT 3.2. *Any method $\mathcal{M}$ is equal to its double dual, i.e., $(\mathcal{M}^*)^* = \mathcal{M}$.*

STATEMENT 3.3. *If a method is explicit, then its dual is also explicit.*

Examples of self-dual methods are:

- Kutta's 3rd-order method [21, p. 440],
- the classic Runge–Kutta method [21, p. 443],
- the $3/8$ rule [21, p. 441],
- the general case of 4-stage methods of order 4 with symmetrically placed nodes [8, equation (5.4d)],
- a special case of 4-stage methods of order 4 [21, p. 442, equation (V)] (see also [8, equation (5.4f)]),
- the Gauss–Legendre methods [17], [2, p. 56],
- Butcher's Lobatto [3, Table 3] and Lobatto IIIC [10] methods.

A map from a method to its dual is an involution on Runge–Kutta methods that exchanges Q- and D-type conditions:

THEOREM 3.4. *Consider a method $\mathcal{M}$ that satisfies $B(l)$, $C(m)$, and $D(n)$ (cf. (2.1)–(2.3)). Then its dual $\mathcal{M}^*$ satisfies $B(l)$, $C(n)$, and $D(m)$.*

Proof. Within the proof, set $I = s+1-i$ and $J = s+1-j$. For any $0 \le k < l$, it holds that

$$\begin{aligned}\boldsymbol{b}^*(\boldsymbol{c}^*)^k &= \sum_{j=1}^{s} b_j^* c_j^{*k} = \sum_{j=1}^{s} b_J (1-c_J)^k = \sum_{J=1}^{s} b_J \sum_{k'=0}^{k} \binom{k}{k'} (-c_J)^{k'} \\ &= \sum_{k'=0}^{k} (-1)^{k'} \binom{k}{k'} \sum_{J=1}^{s} b_J c_J^{k'} = \sum_{k'=0}^{k} (-1)^{k'} \binom{k}{k'} \frac{1}{k'+1} = \frac{1}{k+1}.\end{aligned}$$

Thus, $\mathcal{M}^*$ satisfies $B(l)$. For any $0 \le k < n$, it holds that

$$\begin{aligned}\left(\boldsymbol{A}^*(\boldsymbol{c}^*)^k\right)_i &= \sum_{j=1}^{s} a_{ij}^* c_j^{*k} = \sum_{J=1}^{s} \frac{b_J a_{JI}}{b_I} (1-c_J)^k = \frac{1}{b_I} \sum_{J=1}^{s} b_J a_{JI} \sum_{k'=0}^{k} \binom{k}{k'} (-c_J)^{k'} \\ &= \sum_{k'=0}^{k} (-1)^{k'} \binom{k}{k'} \frac{1}{b_I} \sum_{J=1}^{s} b_J c_J^{k'} a_{JI} = \sum_{k'=0}^{k} (-1)^{k'} \binom{k}{k'} \frac{1-c_I^{k'+1}}{k'+1} \\ &= (1-c_I)^{k+1}/(k+1) = c_i^{*,k+1}/(k+1).\end{aligned}$$

Thus, $\mathcal{M}^*$ satisfies $C(n)$. For any $0 \le k < m$, it holds that

$$\begin{aligned}&\left((\boldsymbol{b}^*.(\boldsymbol{c}^{*\mathrm{T}})^k)\boldsymbol{A}^*\right)_j \\ &\quad= \sum_{i=1}^{s} b_i^* c_i^{*k} a_{ij}^* = \sum_{I=1}^{s} b_I (1-c_I)^k \frac{b_J a_{JI}}{b_I} = b_J \sum_{I=1}^{s} a_{JI} \sum_{k'=0}^{k} \binom{k}{k'} (-c_I)^{k'} \\ &\quad= b_J \sum_{k'=0}^{k} (-1)^{k'} \binom{k}{k'} \sum_{I=1}^{s} a_{JI} c_I^{k'} = b_J \sum_{k'=0}^{k} (-1)^{k'} \binom{k}{k'} \frac{c_J^{k'+1}}{k'+1} \\ &\quad= b_J \left(1-(1-c_J)^{k+1}\right)/(k+1) = b_j^* \left(1-c_j^{*,k+1}\right)/(k+1).\end{aligned}$$

Thus, $\mathcal{M}^*$ satisfies $D(m)$. $\square$

4. SOL and CNC heuristics. The following is virtually the definition of "*stage order*" in [25, Def. 1]:

DEFINITION 4.1. *A stage i is said to be of* strong stage order *at least p, if*

$$q_{n,i} = \boldsymbol{a}_{i*}\boldsymbol{c}^n - c_i^{n+1}/(n+1) = 0 \quad \textit{for all } 0 \le n < p,$$

and, whenever $a_{ij} \neq 0$, then the stage j is of strong stage order at least $p-1$.

The next definition is weaker but provides more flexibility:

DEFINITION 4.2. *A stage i is said to be of* stage order *at least p if*

$$Q_i(\mathrm{t}) = \boldsymbol{a}_{*i}\boldsymbol{\Phi}(\mathrm{t}) - c_i^{|\mathrm{t}|}/\mathrm{t}! = 0 \quad \textit{for all rooted trees } |\mathrm{t}| \textit{ with } |\mathrm{t}| \le p.$$

STATEMENT 4.3. *If a stage is of strong stage order p, then it is also of stage order p.*

The advantage of Definition 4.1 is that one can ensure that a stage is of a certain stage order by only considering the vectors $\boldsymbol{q}_0$, $\boldsymbol{q}_1$, $\boldsymbol{q}_2, \ldots$ instead of $\boldsymbol{Q}(\mathrm{t})$ for arbitrary rooted trees t. For constructing an explicit Runge–Kutta method of high order p, one can apply the Stage Order Layers (SOL) heuristic, which consists in making stages with non-zero weights to be of sufficiently high stage order $q < p$, so that many order conditions of Q-type, $\boldsymbol{b}(\boldsymbol{Q}(\mathrm{t}_1).\boldsymbol{Q}(\mathrm{t}_2).\cdots.\boldsymbol{Q}(\mathrm{t}_k).\boldsymbol{c}^n) = 0$ with $\min\big(|\mathrm{t}_1|, |\mathrm{t}_2|, \ldots, |\mathrm{t}_k|\big) \le q$, are automatically satisfied. If $q \ge \frac{1}{2}p - 1$, then (besides the quadrature conditions) only the conditions with $k = 1$, i.e., $\boldsymbol{b}(\boldsymbol{Q}(\mathrm{t}).\boldsymbol{c}^n) = 0$ with $|\mathrm{t}| + n < p$, need to be dealt with. Sequential stage order layers, i.e., subsets of stages with the same stage order, highly increase the redundancy of the order conditions of Q-type outside of each layer (i.e., at later stages). A prime example of such an approach is the method in [12]; see Figure 6.1.

The following definition may be seen as a dual version of Definition 4.2:

DEFINITION 4.4. *A stage j is said to be of* weak stage co-order *at least p if*

$$D_j(\mathrm{t}) = \big(\boldsymbol{b}.\boldsymbol{\Phi}^{\mathrm{T}}(\mathrm{t})\big)\boldsymbol{a}_{*j} - b_j(1 - c_j^{|\mathrm{t}|})/\mathrm{t}! = 0 \quad \textit{for all rooted trees } \mathrm{t} \textit{ with } |\mathrm{t}| \le p.$$

STATEMENT 4.5. *If a method satisfies $D(1)$, i.e., $\boldsymbol{d}_0 = \boldsymbol{b}\boldsymbol{A} - \boldsymbol{b}.(\boldsymbol{1} - \boldsymbol{c})^{\mathrm{T}} = \boldsymbol{0}$, then all stages are of weak stage co-order at least* 1.

For explicit methods, Definition 4.4 is not practically useful because it is hardly possible to make some non-zero weights stages to be of weak stage co-order higher than 1: Consider an explicit s-stage method with $\boldsymbol{d}_0 = \boldsymbol{0}$. As $b_s \neq 0$ and $d_{0,s} = -b_s(1 - c_s) = 0$, one must have $c_s = 1$.[1] Then $d_{0,s-1} = b_s a_{s,s-1} - b_{s-1}(1 - c_{s-1}) = 0$ implies

$$\begin{aligned} a_{s,s-1} &= b_{s-1}(1 - c_{s-1})/b_s \qquad \text{and} \\ d_{1,s-1} &= b_s a_{s,s-1} - b_{s-1}(1 - c_{s-1}^2)/2 = b_{s-1}(1 - c_{s-1})^2/2. \end{aligned}$$

In order to have the stage $(s-1)$ to be of weak stage co-order at least 2, one must have $a_{s,s-1} = 0$.[2]

[1] If $c_s = 1$, then $D_s(\mathrm{t}) = 0$ for all rooted trees t, i.e., the stage s has infinite weak stage co-order. This is dual to the statement that the stage 1 has infinite stage order, as $c_1 = 0$ and $Q_1(\mathrm{t}) = 0$ for all t.

[2] This is dual to the statement that $q_{1,2} = -c_2^2/2 = 0$ only if $c_2 = a_{21} = 0$.

DEFINITION 4.6. *Let $\mathcal{S} = \{1, 2, \ldots, s\}$ be the set of all stages. A* node cluster *is a triple $\mathcal{C} = (S, Q, D)$, where S is a non-empty subset of $\mathcal{S}$ such that the nodes corresponding to any two stages i, j in S are identical, $c_i = c_j$, and $Q \subseteq \mathbb{R}^{|S|}$ and $D \subseteq (\mathbb{R}^{|S|})^*$ are subspaces (here $|S|$ stands for the cardinality of S, and vectors are indexed by the elements of S) that satisfy the following orthogonality conditions:*

$$Q = \left\{ \boldsymbol{q} \in \mathbb{R}^{|S|} \,\middle|\, \textstyle\sum_{i\in S} b_i q_i = \sum_{i\in S} d_i q_i = 0 \textit{ for all } \boldsymbol{d} \in D \right\},$$
$$D = \left\{ \boldsymbol{d} \in (\mathbb{R}^{|S|})^* \,\middle|\, \textstyle\sum_{i\in S} d_i = \sum_{i\in S} d_i q_i = 0 \textit{ for all } \boldsymbol{q} \in Q \right\},$$

i.e., Q and D are the orthogonal complements of $D + \mathit{span}(\boldsymbol{b}|_S)$ and $Q + \mathit{span}(\mathbf{1}|_S)$, respectively. If $\sum_{i\in S} b_i \neq 0$, then $\mathcal{C}$ is said to be a quadrature cluster.

THEOREM 4.7. *Let $\mathcal{C} = (S, Q, D)$ be a node cluster. If $\mathcal{C}$ is a quadrature cluster, then*

$$\mathbf{1}|_S \notin Q, \quad \boldsymbol{b}|_S \notin D, \quad \textit{and} \quad \dim Q + \dim D = |S| - 1.$$

If $\mathcal{C}$ is a non-quadrature cluster, i.e., $\sum_{i\in S} b_i = 0$, then

$$\mathbf{1}|_S \in Q, \quad \boldsymbol{b}|_S \in D, \quad \textit{and} \quad \dim Q + \dim D = |S|.$$

Proof. As Q is the orthogonal complement of $D + \mathrm{span}(\boldsymbol{b}|_S)$, its dimension is $\dim Q = |S| - \dim\big(D + \mathrm{span}(\boldsymbol{b}|_S)\big)$. If $\boldsymbol{b}|_S \mathbf{1}|_S = \sum_{i\in S} b_i \neq 0$, then $\boldsymbol{b}|_S$ is not in D, so $\dim\big(D + \mathrm{span}(\boldsymbol{b}|_S)\big) = \dim D + 1$. In a quadrature cluster, $\mathbf{1}|_S$ is not orthogonal to $\boldsymbol{b}|_S$ and thus is not in Q. In the case of a non-quadrature cluster, $\sum_{i\in S} b_i = 0$, the row vector $\boldsymbol{b}|_S$ is orthogonal to $Q + \mathrm{span}(\mathbf{1}|_S)$, so $\boldsymbol{b}|_S \in D$ and $\dim\big(D + \mathrm{span}(\boldsymbol{b}|_S)\big) = \dim D$. As the vector $\mathbf{1}|_S$ is orthogonal to both D and $\boldsymbol{b}|_S$, it is in Q. $\square$

DEFINITION 4.8. *A node cluster $\mathcal{C} = (S, Q, D)$ is said to be of* cluster order *at least p if $\boldsymbol{Q}(t)$ restricted to S is in the subspace Q for all rooted trees* t *with* $|\mathrm{t}| \leq p$.

Consider the following two filtrations on the algebra of column vectors $\mathbb{R}^s$ with the product taken elementwise [19, pp. 560 and 561], [20, pp. 683 and 684]:

$$\begin{array}{ccccccccccccc}
\Phi_0 & \subseteq & \Phi_1 & \subseteq & \Phi_2 & \subseteq & \ldots & \subseteq & \Phi_p & \subseteq & \ldots & \subseteq & \mathbb{R}^s \\
\cup & & \cup & & \cup\!| & & & & \cup\!| & & & & \| \\
Q_0 & = & Q_1 & \subseteq & Q_2 & \subseteq & \ldots & \subseteq & Q_p & \subseteq & \ldots & \subseteq & \mathbb{R}^s
\end{array}$$

The subspace $\Phi_p \subseteq \mathbb{R}^s$ is spanned by the vectors $\boldsymbol{\Phi}(\mathrm{t})$ for all rooted trees t with $|\mathrm{t}| \leq p + 1$. For instance, $\Phi_0 = \mathrm{span}(\mathbf{1})$, $\Phi_1 = \mathrm{span}(\mathbf{1}, \boldsymbol{c})$, and $\Phi_2 = \mathrm{span}(\mathbf{1}, \boldsymbol{c}, \boldsymbol{c}^2, \boldsymbol{A}\boldsymbol{c})$. A recursive definition of the subspaces Φ_p is the following: $\Phi_0 = \mathrm{span}(\mathbf{1})$, and Φ_p is generated by the subsets of Φ_{p-1}, $\boldsymbol{A}\Phi_{p-1}$ and elementwise products of the subspaces $\Phi_q.\Phi_{p-q}$, where $0 < q < p$. (For sets X and Y and a binary operation $\star$, the set operation is defined as $X \star Y = \{x \star y \,|\, x \in X \text{ and } y \in Y\}$; also $x \star Y = \{x\} \star Y$.) Similarly, $Q_0 = Q_1 = \{\mathbf{0}\}$, $Q_2 = \mathrm{span}(\boldsymbol{q}_1)$, and Q_p is generated by $\boldsymbol{q}_{p-1}$, $\boldsymbol{A}Q_{p-1}$, and $Q_q.\Phi_{p-q}$, where $1 < q < p$. For a method of order p, the Q-type order conditions can be written as $\boldsymbol{b}Q_{p-1} = \{0\}$.

Consider the following non-strictly increasing sequence of subspaces (with no structure that is related to their elementwise products) of the vector space of row vectors $(\mathbb{R}^s)^*$: $D_0 = \{\mathbf{0}\}$, $D_1 = \mathrm{span}(\boldsymbol{d}_0)$, and D_p is generated by $D_{p-1}.\Phi_1^{\mathrm{T}}$, $D_{p-1}\boldsymbol{A}$, and $\boldsymbol{D}(\mathrm{t})$ for all rooted trees t with $|\mathrm{t}| = p$. For example, if $\boldsymbol{d}_0 = \mathbf{0}$, then

$$D_2 = \mathrm{span}(\boldsymbol{d}_1) \quad \text{and} \quad D_3 = \mathrm{span}\big(\boldsymbol{d}_1, \boldsymbol{d}_1.\boldsymbol{c}^{\mathrm{T}}, \boldsymbol{d}_1\boldsymbol{A}, \boldsymbol{d}_2, (\boldsymbol{b}.(\boldsymbol{A}\boldsymbol{c})^{\mathrm{T}})\boldsymbol{A} - \boldsymbol{b}.(\mathbf{1} - \boldsymbol{c}^3)^{\mathrm{T}}/6\big).$$

For a method of order p, the D-type order conditions can be written as $D_{p-1}\mathbf{1} = \{0\}$ or, equivalently, $D_{q-1}\Phi_{p-q} = \{0\}$, for all $1 < q \leq p$.

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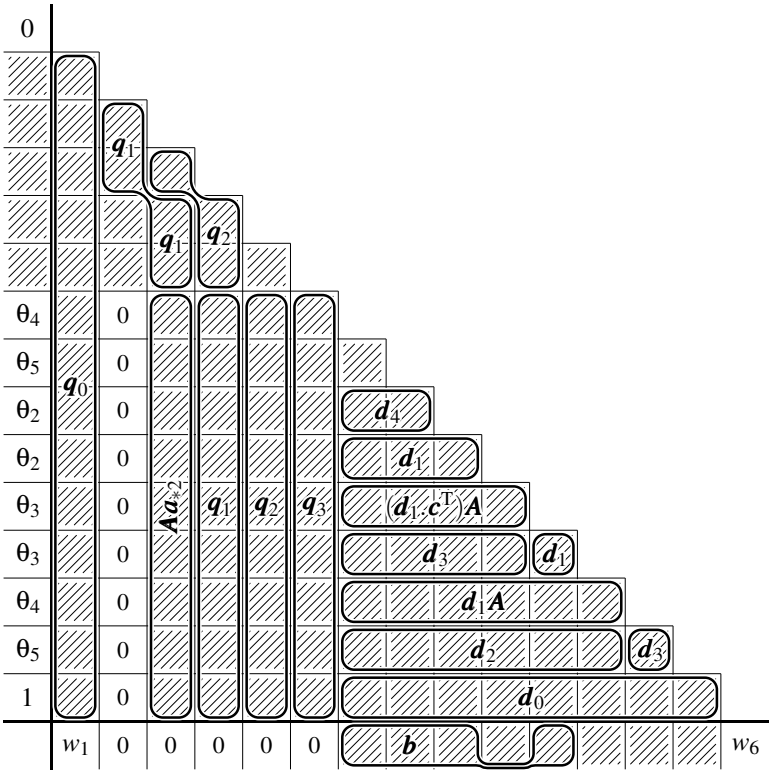

FIG. 4.1. *Butcher tableau [4, p. 191] with all but the final steps of the method's construction. The shaded cells correspond to the nodes, weights, and coefficients that are to be determined.*
The respective entries are found by the following equations:
"$\boldsymbol{b}$": by equation (5.1);
"$\boldsymbol{q}_0$": by equation (5.2);
"$\boldsymbol{q}_1$" and "$\boldsymbol{q}_2$" in the upper left corner: by equation (5.3);
parallel blocks with "$\boldsymbol{A}\boldsymbol{a}_{*2}$", "$\boldsymbol{q}_1$", "$\boldsymbol{q}_2$", and "$\boldsymbol{q}_3$": by equation (5.4);
"$\boldsymbol{d}_0$": by equation (5.5);
"$\boldsymbol{d}_1$": by equation (5.6);
"$\boldsymbol{d}_2$" and "$\boldsymbol{d}_1\boldsymbol{A}$": by equation (5.7);
"$\boldsymbol{d}_3$" and "$(\boldsymbol{d}_1.\boldsymbol{c}^T)\boldsymbol{A}$": by equation (5.8);
"$\boldsymbol{d}_4$": by equation (5.9).

A practical working definition of co-order is:

DEFINITION 4.9. *A node cluster $\mathcal{C} = (S, Q, D)$ is said to be of* cluster co-order *at least p if for any row vector $\boldsymbol{d}$ in the subspace D_p, its restriction to S lies in D.*

The Counterpoised Node Clusters (CNC) heuristic consists in partitioning the set of stages $\mathcal{S}$ into node clusters and making the node clusters order *and* co-order sufficiently high. The word "counterpoised" emphasizes that the subspace D is orthogonal to the vector $\mathbf{1}|_S$ for any node cluster (S, Q, D); that helps to satisfy order conditions such as $\boldsymbol{d}_n\boldsymbol{c}^m = 0$. Notable examples of using repeated nodes to form high order/co-order node clusters are [24, Table 3.3], [11, Table 1], and [14] (see Figure 6.1).

Below, the type of a quadrature/non-quadrature node cluster (S, Q, D) will be written as $\bullet^d_{d^*}$ / $\circ^d_{d^*}$ to indicate the dimensions $d = \dim Q$ and $d^* = \dim D$, or as ${}^p_{p^*}\bullet^d_{d^*}$ / ${}^p_{p^*}\circ^d_{d^*}$ to additionally indicate its cluster order p and cluster co-order p^*.

For a node cluster with only one stage, the possible types are $\bullet_0^0$ and $\circ_0^1$. As $\dim D = 0$ and the dimension of Q is uniquely determined by whether the cluster is a quadrature one or not, the dimensions of Q and D are omitted in one node cluster in Figure 6.1. With two stages, the types can be $\bullet_1^0$, $\bullet_0^1$, $\circ_1^1$, and $\circ_0^2$. For a node cluster of type $\bullet_1^1$, there is an interesting possibility: $Q = \operatorname{span}\big([\ g_1 \quad g_2 \quad g_3\ ]^{\mathrm{T}}\big)$, $D = \operatorname{span}\big([\ (g_2 - g_3) \quad (g_3 - g_1) \quad (g_1 - g_2)\ ]\big)$, and $\boldsymbol{b}|_S \in \operatorname{span}\big([\ (g_2 - g_3)/g_1 \quad (g_3 - g_1)/g_2 \quad (g_1 - g_2)/g_3\ ]\big)$. Not only $\boldsymbol{b}|_S Q = \{0\}$, but $\boldsymbol{b}|_S(Q.Q) = \{0\}$, too.

5. A family of methods of order 10. A Runge–Kutta method of order 10 satisfies the order conditions $\boldsymbol{b}\boldsymbol{\Phi}(\mathrm{t}) = 1/\mathrm{t}!$ for the 1205 rooted trees t such that $|\mathrm{t}| \le 10$. For a tree $\mathrm{t} = [\mathrm{t}_1\mathrm{t}_2 \ldots \mathrm{t}_n]$, the corresponding integer partition is $|\mathrm{t}| - 1 = |\mathrm{t}_1| + |\mathrm{t}_2| + \ldots + |\mathrm{t}_n|$; see [8, pp. 50 and 65]. The partition for $\mathrm{t}_1 \cdot \mathrm{t}_2 \cdot \ldots \cdot \mathrm{t}_n$ is the sum of the partitions for $\mathrm{t}_1, \mathrm{t}_2, \ldots, \mathrm{t}_n$.

The explicit methods constructed below are based on the 6-points Lobatto quadrature:

$$\int_0^1 \mathrm{d}\theta\, f(\theta) \approx \sum_{k=1}^{6} w_k f(\theta_k), \qquad \alpha, \beta = \sqrt{\tfrac{1}{21}(7 \pm 2\sqrt{7})},$$
$$\theta_1 = 0, \qquad \theta_{2,5} = \tfrac{1}{2}(1 \mp \alpha), \qquad \theta_{3,4} = \tfrac{1}{2}(1 \mp \beta), \qquad \theta_6 = 1,$$
$$w_1 = w_6 = \tfrac{1}{30}, \quad w_2 = w_5 = \tfrac{1}{60}(14 - \sqrt{7}), \quad w_3 = w_4 = \tfrac{1}{60}(14 + \sqrt{7}).$$

The column vector of nodes $\boldsymbol{c}$ is chosen by setting

$$c_1 = \theta_1 = 0, \qquad c_7 = c_{13} = \theta_4, \qquad c_8 = c_{14} = \theta_5,$$
$$c_9 = c_{10} = \theta_2, \qquad c_{11} = c_{12} = \theta_3, \quad \text{and} \quad c_{15} = \theta_6 = 1.$$

To satisfy the quadrature order conditions $\boldsymbol{b}\boldsymbol{\Phi}\big([\bullet^n]\big) = \boldsymbol{b}\boldsymbol{c}^n = 1/(n+1)$ corresponding to the partitions $n = 1 + 1 + \ldots + 1$, where $0 \le n < 10$, the row vector of weights $\boldsymbol{b}$ is determined by $b_1 = w_1 = b_{15} = w_6$, $b_j = 0$, for all $2 \le j \le 6$, and

$$b_9 = w_2 - b_{10}, \qquad b_{11} = w_3 - b_{12}, \qquad b_7 = w_4 - b_{13}, \qquad b_8 = w_5 - b_{14}. \tag{5.1}$$

The first column $\boldsymbol{a}_{*1}$ is determined from $\boldsymbol{q}_0 = \boldsymbol{A}\boldsymbol{1} - \boldsymbol{c} = \boldsymbol{0}$:

$$a_{i1} = c_i - \sum_{j=2}^{i-1} a_{ij}, \qquad 2 \le i \le 15. \tag{5.2}$$

The coefficients a_{32}, a_{42}, a_{43}, a_{53}, a_{54}, a_{63}, and a_{64} are found from $\boldsymbol{q}_1 = -\frac{1}{2}c_2^2\boldsymbol{e}_2$ and

by increasing the redundancy of the order conditions relation $\boldsymbol{q}_2 \in \operatorname{span}(\boldsymbol{q}_1, \boldsymbol{A}\boldsymbol{q}_1)$:

$$
\begin{aligned}
a_{32} &= \frac{c_3^2}{2c_2},\\
a_{42} &= \frac{c_4^2(3c_3 - 2c_4)}{2c_2c_3},\\
a_{43} &= \frac{c_4^2(c_4 - c_3)}{c_3^2},\\
a_{53} &= \frac{c_5^2(3c_4 - 2c_5) - a_{52}c_2(6c_4 - 4c_3)}{6c_3(c_4 - c_3)},\\
a_{54} &= \frac{c_5^2(2c_5 - 3c_3) + 2a_{52}c_2c_3}{6c_4(c_4 - c_3)},\\
a_{63} &= \frac{c_6^2(3c_4 - 2c_6) - a_{62}c_2(6c_4 - 4c_3) + 6a_{65}c_5(c_5 - c_4)}{6c_3(c_4 - c_3)},\\
a_{64} &= \frac{c_6^2(2c_6 - 3c_3) + 2a_{62}c_2c_3 - 6a_{65}c_5(c_5 - c_3)}{6c_4(c_4 - c_3)}.
\end{aligned}
\tag{5.3}
$$

The coefficients a_{ij}, where $7 \le i \le 15$ and $3 \le j \le 6$, are found from the conditions $(\boldsymbol{A}^2\boldsymbol{q}_1)_i = -\frac{1}{2}c_2^2(\boldsymbol{A}\boldsymbol{a}_{*2})_i = 0$ and $q_{1,i} = q_{2,i} = q_{3,i} = 0$:

$$
\begin{bmatrix} a_{i3}\\ a_{i4}\\ a_{i5}\\ a_{i6} \end{bmatrix}
=
\begin{bmatrix}
a_{32} & a_{42} & a_{52} & a_{62}\\
c_3 & c_4 & c_5 & c_6\\
c_3^2 & c_4^2 & c_5^2 & c_6^2\\
c_3^3 & c_4^3 & c_5^3 & c_6^3
\end{bmatrix}^{-1}
\begin{bmatrix}
0\\
\frac{1}{2}c_i^2 - \sum_{j=7}^{i-1} a_{ij}c_j\\
\frac{1}{3}c_i^3 - \sum_{j=7}^{i-1} a_{ij}c_j^2\\
\frac{1}{4}c_i^4 - \sum_{j=7}^{i-1} a_{ij}c_j^3
\end{bmatrix}.
\tag{5.4}
$$

This makes $\boldsymbol{q}_1 = -\frac{1}{2}c_2^2\boldsymbol{e}_2$, $\boldsymbol{A}\boldsymbol{q}_1 = -\frac{1}{2}c_2^2\boldsymbol{a}_{*2}$, and $\boldsymbol{q}_2 = c_2(c_2 - \frac{2}{3}c_3)\boldsymbol{a}_{*2} - \frac{1}{3}c_2^3\boldsymbol{e}_2$. Also, $\boldsymbol{b}.\boldsymbol{Q}^{\mathrm{T}}(\mathrm{t}) = \boldsymbol{0}$ for any rooted tree t with $|\mathrm{t}| \le 4$, or any stage i with $b_i \ne 0$ is of stage order at least 4. The order conditions with partitions containing the parts 2, 3, and 4 are satisfied if those ones are where these parts are decomposed into $1+1$, $1+1+1$, and $1+1+1+1$, respectively. For any i, the coefficient a_{ij}, where $j \le 6$, is now expressed through c_2, c_3, c_4, c_5, c_6, a_{52}, a_{62}, a_{65}, and the coefficients a_{ik} with $k \ge 7$.

The dependence of the coefficients matrix $\boldsymbol{A}$ on the node c_2 is inconsequential: Due to $\boldsymbol{q}_0 = \boldsymbol{0}$ and $\boldsymbol{q}_1 = -\frac{1}{2}c_2^2\boldsymbol{e}_2$, only the first and second columns $\boldsymbol{a}_{*1}$ and $\boldsymbol{a}_{*2}$ depend on c_2. The column vector $\boldsymbol{a}_{*1} + \boldsymbol{a}_{*2}$ depends on c_2 only in its second component. The second column $\boldsymbol{a}_{*2}$ is inversely proportional to c_2.

As $c_{15} = 1$ and $\boldsymbol{a}_{*,15} = \boldsymbol{0}$, one has $d_{n,15} = 0$ for all n. The coefficients $a_{15,j}$, where $7 \le j \le 14$, are determined from $d_{0,j} = 0$:

$$
a_{15,j} = \frac{1}{w_6}\left(b_j(1 - c_j) - \sum_{i=j+1}^{14} b_i a_{ij}\right), \qquad 7 \le j \le 14.
\tag{5.5}
$$

Due to $b_2 = 0$, $\boldsymbol{b}.\boldsymbol{a}_{*2}^{\mathrm{T}} = \mathbf{0}$, and $\boldsymbol{b}.(\boldsymbol{A}\boldsymbol{a}_{*2})^{\mathrm{T}} = \mathbf{0}$, one has $d_{n,2} = 0$ and $\boldsymbol{d}_n\boldsymbol{a}_{*2} = 0$, for all n. For $0 \le m \le 3$,

$$\begin{aligned}\boldsymbol{d}_n\boldsymbol{c}^m &= (\boldsymbol{b}.\boldsymbol{c}^{n\mathrm{T}})\boldsymbol{A}\boldsymbol{c}^m - \boldsymbol{b}\frac{\boldsymbol{c}^m - \boldsymbol{c}^{m+n+1}}{n+1}\\ &= (\boldsymbol{b}\boldsymbol{c}^{n\mathrm{T}})\boldsymbol{q}_m + \boldsymbol{b}\frac{\boldsymbol{c}^{m+n+1}}{m+1} - \frac{1}{(n+1)(m+1)} + \frac{1}{(n+1)(m+n+2)}\\ &= 0 + \frac{1}{(m+1)(m+n+2)} + \frac{-(m+n+2)+(m+1)}{(n+1)(m+1)(m+n+2)} = 0,\end{aligned}$$

for all $n \le 8 - m$. Thus, equation (5.5) ensures that $\boldsymbol{d}_0 = \mathbf{0}$ and makes the order conditions corresponding to the partitions $n = n$, where $2 \le n \le 9$, satisfied if the ones with the partitions $n - 1 = (n-1)$ and $n = (n-1) + 1$ are.

The remaining order conditions yet to be satisfied can be written as conditions of D-type, $\boldsymbol{d}_n\Phi_m = \{0\}$, with $1 \le n \le 4$ and $n + m \le 9$. They correspond to partitions

$$\begin{array}{llll} 6 = 5+1, & 7 = 6+1, & 8 = 7+1, & 9 = 8+1,\\ 7 = 5+1+1, & 8 = 6+1+1, & 9 = 7+1+1, & 8 = 5+1+1+1,\\ 9 = 6+1+1+1, & 9 = 5+1+1+1+1. & & \end{array}$$

To absorb non-zero values of, e.g., $d_{1,14} = w_6 a_{15,14} - b_{14}(1-\theta_5^2)/2 = b_{14}(1-\theta_5)^2/2$, the stages are lumped into node clusters $S_4 = \{7, 13\}$, $S_5 = \{8, 14\}$, $S_2 = \{9, 10\}$, and $S_3 = \{11, 12\}$ of type ${}^4_4\bullet^0_1$. The cluster order 4 with $Q = \{\mathbf{0}\}$ is already achieved, as all the stages from 7 to 15 are of stage order 4. For the cluster co-order to be at least 4, the vectors $\boldsymbol{d}_1$, $\boldsymbol{d}_2$, $\boldsymbol{d}_1\boldsymbol{A}$, $\boldsymbol{d}_3$, $(\boldsymbol{d}_1.\boldsymbol{c}^{\mathrm{T}})\boldsymbol{A}$, $\boldsymbol{d}_1\boldsymbol{A}^2$, and $\boldsymbol{d}_2\boldsymbol{A}$ restricted to any of these four node clusters should be proportional to the row vector $[\,-1\ \ 1\,]$. The coefficients $a_{10,7}$, $a_{10,8}$, $a_{10,9}$, and $a_{12,11}$ are found from $d_{1,7} + d_{1,13} = 0$, $d_{1,8} + d_{1,14} = 0$, $d_{1,9} + d_{1,10} = 0$, and $d_{1,11} + d_{1,12} = 0$, respectively:

$$\begin{aligned} a_{10,7} &= \frac{1}{b_{10}(1-\theta_2)}\Bigg(\tfrac{1}{2}w_4(1-\theta_4)^2 - \sum_{\substack{i=8\\ i\neq 10}}^{14} b_i(1-c_i)(a_{i,7}+a_{i,13})\Bigg),\\ a_{10,8} &= \frac{1}{b_{10}(1-\theta_2)}\Bigg(\tfrac{1}{2}w_5(1-\theta_5)^2 - \sum_{\substack{i=9\\ i\neq 10}}^{14} b_i(1-c_i)(a_{i,8}+a_{i,14})\Bigg),\\ a_{10,9} &= \frac{1}{b_{10}(1-\theta_2)}\Bigg(\tfrac{1}{2}w_2(1-\theta_2)^2 - \sum_{i=11}^{14} b_i(1-c_i)(a_{i,9}+a_{i,10})\Bigg),\\ a_{12,11} &= \frac{1}{b_{12}(1-\theta_3)}\Bigg(\tfrac{1}{2}w_3(1-\theta_3)^2 - \sum_{i=13}^{14} b_i(1-c_i)(a_{i,11}+a_{i,12})\Bigg). \end{aligned} \tag{5.6}$$

The row vector $\boldsymbol{d}_1$ now has the following structure:

$$\boldsymbol{d}_1 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & -d_{1,13} & -d_{1,14} & -d_{1,10} & d_{1,10} & -d_{1,12} & d_{1,12} & d_{1,13} & d_{1,14} & 0 \end{bmatrix}.$$

For an order condition $\boldsymbol{b}\boldsymbol{\Phi}(\mathrm{t}) = 1/\mathrm{t}!$, the partition of t will be called a $\boldsymbol{b}$-partition. For a rooted tree $\mathrm{t} = [\mathrm{t}_1\mathrm{t}_2\ldots\mathrm{t}_m]$, with $|\mathrm{t}| \le 9 - n$, the order condition $\boldsymbol{d}_n\boldsymbol{\Phi}(\mathrm{t}) = 0$, which is related to $\boldsymbol{b}\boldsymbol{\Phi}\big([\mathrm{t}\bullet^n]\big) = 1/[\mathrm{t}\bullet^n]! = 1/(|\mathrm{t}| + n + 1)\mathrm{t}!$ and corresponds to a $\boldsymbol{b}$-partition $|\mathrm{t}| + n = |\mathrm{t}| + 1 + 1 + \ldots + 1$ can be further classified by $|\mathrm{t}| - 1 = |\mathrm{t}_1| + |\mathrm{t}_2| + \ldots + |\mathrm{t}_m|$, which

will be referred to as a $\boldsymbol{d}_n$-partition. For any tree t with $|\mathrm{t}| \leq 4$, the condition $\boldsymbol{d}_1.\boldsymbol{Q}^{\mathrm{T}}(\mathrm{t}) = \boldsymbol{0}$ holds. Within the order conditions $\boldsymbol{d}_1\Phi_8 = \{0\}$, only those with $\boldsymbol{d}_1$-partitions having 1, 5, 6, or 7 as parts, need to be verified. Due to the structure of the row vector $\boldsymbol{d}_1$, the condition $\boldsymbol{d}_1\boldsymbol{c}^n = 0$ with the $\boldsymbol{d}_1$-partition $n = 1 + 1 + \ldots + 1$ (the analogue of a quadrature condition, but for $\boldsymbol{d}_1$) is satisfied for all n.

The coefficients $a_{14,j}$ and $a_{13,j}$, where $7 \leq j \leq 12$, are found from increasing the redundancy of the order condition relations

$$\boldsymbol{d}_2 = \gamma_{20}\boldsymbol{d}_1 + \gamma_{21}(\boldsymbol{d}_1.\boldsymbol{c}^{\mathrm{T}}), \qquad \boldsymbol{d}_1\boldsymbol{A} = \gamma_{a0}\boldsymbol{d}_1 + \gamma_{a1}(\boldsymbol{d}_1.\boldsymbol{c}^{\mathrm{T}}), \tag{5.7}$$

taken from the 7th to 12th components. The four constants γ_{20}, γ_{21}, γ_{a0}, and γ_{a1} are found from these two relations taken from the 13th and 14th components. After determining $a_{14,j}$ and $a_{13,j}$, where $7 \leq j \leq 12$, in this way the first six components of both $\boldsymbol{d}_2$ and of $\boldsymbol{d}_1\boldsymbol{A}$ are equal to zero. As $\boldsymbol{d}_0 = 0$, one has $D_1 = \{\boldsymbol{0}\}$, $D_2 = \mathrm{span}(\boldsymbol{d}_1)$, and these relations make D_3 being just the 2-dimensional $\mathrm{span}\big(\boldsymbol{d}_1, \boldsymbol{d}_1.\boldsymbol{c}^{\mathrm{T}}\big)$. The order conditions corresponding to the $\boldsymbol{b}$-partitions $n+2 = (n)+1+1$ are now satisfied if the ones with the partitions $n+1 = (n)+1$ and $n + 2 = (n + 1) + 1$ are. Also, the order conditions with the $\boldsymbol{d}_1$-partitions $n = n$, where $5 \leq n \leq 7$, are satisfied if the ones with $n - 1 = (n - 1)$ and $n = (n - 1) + 1$ are.

The coefficients $a_{11,j}$ and $a_{12,j}$, where $7 \leq j \leq 10$, and the coefficient $a_{14,13}$ are found from increasing the redundancy of the order condition relations

$$\begin{aligned} \boldsymbol{d}_3 &= \gamma_{30}\boldsymbol{d}_1 + \gamma_{31}(\boldsymbol{d}_1.\boldsymbol{c}^{\mathrm{T}}) + \gamma_{32}(\boldsymbol{d}_1.\boldsymbol{c}^{2\mathrm{T}}), \\ (\boldsymbol{d}_1.\boldsymbol{c}^{\mathrm{T}})\boldsymbol{A} &= \gamma_{c0}\boldsymbol{d}_1 + \gamma_{c1}(\boldsymbol{d}_1.\boldsymbol{c}^{\mathrm{T}}) + \gamma_{c2}(\boldsymbol{d}_1.\boldsymbol{c}^{2\mathrm{T}}), \end{aligned} \tag{5.8}$$

taken from the 7th to 10th components. The six constants γ_{30}, γ_{31}, γ_{32}, γ_{c0}, γ_{c1}, and γ_{c2} are found from these two relations taken from the 12th, 13th, and 14th components. The two remaining equations, at the 11th component, are satisfied by tuning the coefficient $a_{14,13}$ and having $c_{11} = c_{12}$. (In the 11-stage methods [24, Table 3.3, p. 74a] and [11, Table 1] of order 8, the stages 5, 6, and 7 are of stage order 3, while the stages 8, 9, 10, and 11 are of stage order 4. In both methods $c_7 = c_8$, which is an essential element of the design. Satisfying some of the conditions of D-type through the choice of nodes is dual to increasing a stage order by setting a node value; see, e.g., [12, Equations (6.1), (6.2), (6.3) and (6.4)].)

The first six components of both $\boldsymbol{d}_3$ and of $(\boldsymbol{d}_1.\boldsymbol{c}^{\mathrm{T}})\boldsymbol{A}$ are now equal to zero, the subspace $D_4 = \mathrm{span}\big(\boldsymbol{d}_1, \boldsymbol{d}_1.\boldsymbol{c}^{\mathrm{T}}, \boldsymbol{d}_1.\boldsymbol{c}^{2\mathrm{T}}\big)$ is 3-dimensional, and the four node clusters based on the S_2, S_3, S_4, and S_5 subsets are of cluster co-order 4. The order conditions corresponding to the partitions $8 = 5 + 1 + 1 + 1$ and $9 = 6 + 1 + 1 + 1$ are now satisfied if the ones with the partitions $6 = 5+1$, $7 = 6+1$, $8 = 7+1$, and $9 = 8+1$ are. Due to equation (5.8), verifying the order conditions with the $\boldsymbol{d}_1$-partitions $6 = 5 + 1$ and $7 = 6 + 1$ is reduced to examining the ones with $4 = 4$, $5 = 4 + 1$, $6 = 4 + 1 + 1$, and $5 = 5$, $6 = 5 + 1$, $7 = 5 + 1 + 1$, respectively.

The order conditions yet to be satisfied can be written as $(\boldsymbol{d}_1.\boldsymbol{c}^{2\mathrm{T}})\boldsymbol{A}\Phi_4 = \{0\}$ and $\boldsymbol{d}_4\Phi_4 = \{0\}$. They correspond to a $\boldsymbol{d}_1$-partition $7 = 5 + 1 + 1$ and a $\boldsymbol{d}_4$-partitions $4 = 1 + 1 + 1 + 1$, $4 = 2 + 1 + 1$, $4 = 2 + 2$, $4 = 3 + 1$, $4 = 4$, respectively, with the corresponding $\boldsymbol{b}$-partitions $9 = 8 + 1$ and $9 = 5 + 1 + 1 + 1 + 1$. As $\boldsymbol{q}_1 = -\frac{1}{2}c_2^2\boldsymbol{e}_2$ and $d_{4,2} = 0$, the $\boldsymbol{d}_4$-partitions containing a part 2 are reduced to the ones where it is decomposed into $1 + 1$.

The coefficients a_{97} and a_{98} are found from increasing the redundancy of the order conditions relation

$$\boldsymbol{d}_4 = \gamma_{40}\boldsymbol{d}_1 + \gamma_{41}(\boldsymbol{d}_1.\boldsymbol{c}^{\mathrm{T}}) + \gamma_{42}(\boldsymbol{d}_1.\boldsymbol{c}^{2\mathrm{T}}) + \gamma_{43}(\boldsymbol{d}_1.\boldsymbol{c}^{3\mathrm{T}}) + \gamma_{4c}(\boldsymbol{d}_1.\boldsymbol{c}^{2\mathrm{T}})\boldsymbol{A}, \tag{5.9}$$

θ_5		a_{87}				
θ_2		$\alpha'_{24}+u'_2a_{87}$	a_{98}			
θ_2	$\cdots$	$a_{97}+\frac{w_4}{b_{10}}(\alpha_{24}+u_2a_{87})$	$a_{98}+\frac{w_5}{b_{10}}\alpha_{25}$	$\frac{w_2}{b_{10}}\alpha_{22}$		
θ_3		$\alpha'_{34}+u'_3a_{87}$	$a_{11,8}$	$\alpha'_{32}-a_{11,10}$		
θ_3		$a_{11,7}+\frac{w_4}{b_{12}}(\alpha_{34}+u_3a_{87})$	$a_{11,8}+\frac{w_5}{b_{12}}\alpha_{35}$	$\alpha'_{32}+\frac{w_2}{b_{12}}\alpha_{32}-a_{12,10}$		
θ_4	$\cdots$	$\frac{w_4}{b_{13}}(\alpha_{44}+u_4a_{87})$	$\frac{w_5}{b_{13}}\alpha_{45}$	$\frac{w_2}{b_{13}}\alpha_{42}-a_{13,10}$	$\cdots$	
θ_5		$a_{87}+\frac{w_4}{b_{14}}(\alpha_{54}+u_5a_{87})-a_{14,13}$	$\frac{w_5}{b_{14}}\alpha_{55}$	$\frac{w_2}{b_{14}}\alpha_{52}-a_{14,10}$		
1		$\frac{w_4}{w_6}(\alpha_{64}+u_6a_{87})-a_{15,13}$	$\frac{w_5}{w_6}\alpha_{65}-a_{15,14}$	$\frac{w_2}{w_6}\alpha_{62}-a_{15,10}$		
	$\cdots$	w_4-b_{13}	w_5-b_{14}	w_2-b_{10}	$\cdots$	

θ_3		$\frac{b_{10}}{w_3}\mathrm{A}_{11,10}$					
θ_3		$a_{11,10}+\frac{b_{10}}{b_{12}}\mathrm{A}_{12,10}$	$\frac{w_3}{b_{12}}\alpha_{33}$				
θ_4	$\cdots$	$\frac{b_{10}}{b_{13}}\mathrm{A}_{13,10}$	$\frac{w_3}{b_{13}}\alpha_{43}-a_{13,12}$	$\frac{b_{12}}{b_{13}}\mathrm{A}_{13,12}$			
θ_5		$\frac{b_{10}}{b_{14}}\mathrm{A}_{14,10}$	$\frac{w_3}{b_{14}}\alpha_{53}-a_{14,12}$	$\frac{b_{12}}{b_{14}}\mathrm{A}_{14,12}$	$\frac{b_{13}}{b_{14}}\mathrm{A}_{14,13}$		
1		$\frac{b_{10}}{w_6}\mathrm{A}_{15,10}$	$\frac{w_3}{w_6}\alpha_{63}-a_{15,12}$	$\frac{b_{12}}{w_6}\mathrm{A}_{15,12}$	$\frac{b_{13}}{w_6}\mathrm{A}_{15,13}$	$\frac{b_{14}}{w_6}\mathrm{A}_{15,14}$	
	$\cdots$	b_{10}	w_3-b_{12}	b_{12}	b_{13}	b_{14}	$\cdots$

FIG. 5.1. *Bottom right corner of the Butcher tableau obtained by satisfying the quadrature conditions equation (5.1) and increasing the redundancy in the order conditions of the D-type relations according to equations (5.5), (5.6), (5.7), (5.8), and (5.9). The weights b_{10}, b_{12}, b_{13}, and b_{14} are free parameters. The coefficient a_{87} is determined later.*

taken at the 7th and 8th components. The five constants γ_{40}, γ_{41}, γ_{42}, γ_{43}, and γ_{4c} are found from this relation from the 10th to 14th components. The remaining equation is satisfied by having $c_9 = c_{10}$. The row vectors $\boldsymbol{d}_4$ and $(\boldsymbol{d}_1.\boldsymbol{c}^{2\mathrm{T}})\boldsymbol{A}$ have their second component being equal to zero, but since, for instance, $d_{4,8} + d_{4,14} \neq 0$, their first and the third to sixth components can be non-zero. Nevertheless, the relation equation (5.9) is satisfied at all the fifteen components.

It is possible to construct explicit 15-stage Runge–Kutta methods of order 10 with a different permutation of the 6-points Lobatto quadrature nodes. For the design presented here, it is necessary that the nodes c_{10}, c_{12}, c_{13}, and c_{14} are a permutation of the four interior nodes, that $c_9 = c_{10}$ and $c_{11} = c_{12}$, and that the stages from 7 to 14 use each interior node twice. The nodes in the 6-points Lobatto quadrature are elements of the algebraic extension $\mathbb{Q}(\alpha,\beta)$ of the field of rational numbers $\mathbb{Q}$. Any element of $\mathbb{Q}(\alpha,\beta)$ can be expressed as a linear combination $\xi_1+\xi_2\sqrt{3}+\xi_3\sqrt{7}+\xi_4\sqrt{21}+\xi_5\alpha+\xi_6\beta+\xi_7\sqrt{7}\alpha+\xi_8\sqrt{7}\beta$ with rational weights ξ_i, $1 \leq i \leq 8$. The fifteen constants γ_{20}, γ_{21}, γ_{a0}, γ_{a1}, γ_{30}, γ_{31}, γ_{32}, γ_{c0}, γ_{c1}, γ_{c2}, γ_{40}, γ_{41}, γ_{42}, γ_{43}, and γ_{4c} do not depend on b_{10}, b_{12}, b_{13}, and b_{14} and are in $\mathbb{Q}(\alpha,\beta)$.

All coefficients a_{ij} are now expressed through c_2, c_3, c_4, c_5, c_6, b_{10}, b_{12}, b_{13}, b_{14}, a_{52}, a_{62}, a_{65}, and a_{87}. The structure of the bottom right corner of the Butcher tableau, i.e., the coefficients a_{ij}, for $j \geq 7$, is displayed in Figure 5.1. The forty two constants $\mathrm{A}_{15,14}$, $\mathrm{A}_{15,13}$,

$\mathrm{A}_{14,13}, \ldots, \alpha_{32}, \alpha'_{32}, \alpha_{22}, \ldots, u_3, u'_3, u_2$, and u'_2 are in $\mathbb{Q}(\alpha, \beta)$. When dealing with the order conditions of D-type, in order to at least partially eliminate the presence of the weights, it is convenient to use variables renormalized by weights, $a_{ij} = b_j \mathrm{A}_{ij}/b_i$ and $d_{n,j} = b_j \Delta_{nj}$:

$$\mathrm{A}_{15,j} = 1 - c_j \; - \sum_{i=j+1}^{14} \mathrm{A}_{ij}, \qquad \Delta_{nj} = \mathcal{D}_n(c_j) \; - \sum_{i=j+1}^{14} (1 - c_i^n)\mathrm{A}_{ij},$$

where $\mathcal{D}_n(\theta) = 1 - \theta - (1 - \theta^{n+1})/(n+1)$, e.g., $\mathcal{D}_0(\theta) = 0$ and $\mathcal{D}_1(\theta) = \frac{1}{2}(1-\theta)^2$. The coefficients A_{ij} are those of a "would-be" dual method (as $b_2 = b_3 = b_4 = b_5 = b_6 = 0$, the dual method does not exist).

The four remaining order conditions to be satisfied are $\boldsymbol{d}_4\boldsymbol{c}^4 = \boldsymbol{d}_4(\boldsymbol{c}.\boldsymbol{a}_{*2}) = \boldsymbol{d}_4\boldsymbol{A}\boldsymbol{a}_{*2} = \boldsymbol{d}_4\boldsymbol{q}_3 = 0$. By dimension count, satisfying them would reduce the number of free parameters by four, resulting in a 9-dimensional family of methods of order 10. However, satisfying the remaining conditions with maximal possible generality is cumbersome. One way to simplify the further analysis is to set $c_3 = \theta_3$, $c_4 = \theta_4$, $c_5 = \theta_5$, $c_6 = \theta_2$. Then it is possible to construct a 5-dimensional family of explicit 15-stage methods of order 10, parametrized by c_2, b_{10}, b_{12}, b_{13}, and b_{14}, with coefficients in a certain quadratic extension of $\mathbb{Q}(\alpha, \beta)$.

A more sensible approach would be to increase the number of stage order layers in the opening stages. Let $c_3 = \frac{2}{3}c_4$ (which implies $a_{42} = 0$) and $a_{52} = a_{62} = 0$. With $\boldsymbol{a}_{*2} = a_{32}\boldsymbol{e}_3$, the coefficients a_{i3}, where $7 \le i \le 15$, are all zero. The stages 2, 3, from 4 to 6, and from 7 to 15 are of strong stage order 1, 2, 3, and 4, respectively. With $d_{4,3} = 0$, the condition $\boldsymbol{d}_4(\boldsymbol{c}.\boldsymbol{a}_{*2}) = 0$ is satisfied. The row vector $\boldsymbol{d}_4$ does not depend on a_{65}. The coefficient a_{87} is found from the condition $\boldsymbol{d}_4\boldsymbol{c}^4 = 0$ and is now expressed through c_4, c_5, and c_6. The coefficient a_{65} is found from $\boldsymbol{d}_4\boldsymbol{A}\boldsymbol{a}_{*2} = 0$. The last remaining order condition $\boldsymbol{d}_4\boldsymbol{q}_3 = 0$ is satisfied by setting the node c_6 as follows:

$$c_6 = \frac{U(c_4 + c_5) + 14U'c_4c_5 + U''c_4c_5(c_4 + c_5)}{3U + 14U'(c_4 + c_5) + 2U''(c_4^2 + c_5^2) + 7c_4c_5\big(V + 20V'(c_4 + c_5) + 60V''c_4c_5\big)},$$

with the six constants U, U', U'', V, V', and V'' being in $\mathbb{Q}(\alpha, \beta)$. The result is a 7-dimensional family of explicit 15-stage Runge–Kutta methods of order 10, parametrized by c_2, c_4, c_5, b_{10}, b_{12}, b_{13}, and b_{14}. The following choice of parameters gives a method with comparatively low magnitude of the coefficients (the exact values for its nodes $\boldsymbol{c}$, weights $\boldsymbol{b}$, and coefficients $\boldsymbol{A}$ are available at doi:10.5281/zenodo.15429897):

$$c_2 = \tfrac{2}{15}, \quad c_4 = \tfrac{2}{5}, \quad c_5 = \tfrac{4}{7}, \quad b_{10} = \tfrac{2}{7}w_2, \quad b_{12} = \tfrac{2}{9}w_3, \quad b_{13} = w_4, \quad b_{14} = w_5. \tag{5.10}$$

Under the condition $c_4 + c_5 = 3(1+\beta)/4$, the expression for the parameter c_6 simplifies to $c_6 = U_1/\big(U_2 - (c_5 - c_4)^2\big)$ for some constants U_1 and U_2.

6. Properties of some methods of order 10. The basic properties of some known explicit Runge–Kutta methods of order 10 and of the new method in equation (5.10) are compared in Table 6.1 and in Figures 6.1, 6.2, and 6.3, where the methods are named as follows: C10 is [12], H10 is [14], O10 is [22], F10 is [13], and Z10 is [27].

In C10, the stages from 2 to 11 are forming five stage order layers $\{2\}$, $\{3\}$, $\{4, 5\}$, $\{6, 7\}$, and $\{8, 9, 10, 11\}$; see Figure 6.1, top left panel. To absorb the non-zero values of, e.g., $d_{1,17}$, virtually out of necessity, two counterpoised node clusters $\{14, 17\}$ and $\{12, 16\}$ of type ${}^{6}_{3}\bullet^{0}_{1}$ are used. Still, much effort is spent in the opening for the later stages (from 12 to 18) to be of stage order 6.

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TABLE 6.1

A comparison of six explicit s-stage Runge–Kutta methods of order 10*. The error coefficients are defined as* $T_p^2 = \sum_{\mathrm{t},\ |\mathrm{t}|=p} \big(\boldsymbol{b}\boldsymbol{\Phi}(\mathrm{t}) - 1/\mathrm{t}!\big)^2/\sigma^2(\mathrm{t}) = (1/p!)^2 \sum_{\mathrm{t},\ |\mathrm{t}|=p} \alpha^2(\mathrm{t})\big(\mathrm{t}!\,\boldsymbol{b}\boldsymbol{\Phi}(\mathrm{t}) - 1\big)^2$*, where* $\sigma(\mathrm{t})$ *is the order of the symmetry group of the tree* t *and* $\alpha(\mathrm{t})$ *is the number of monotonic labelings of* t *(see, e.g., [7, Sections 304 and 318], [16, pp. 147 and 158], [8, pp. 58 and 60], [15, pp. 57 and 58]).*

The column labeled as $\min_j b_j$ *shows the minimal value of a non-zero weight. The interval of absolute stability* $[z_R, 0]$ *is the connected component of* $\{z | z \in \mathbb{R} \text{ and } |R(z)| \le 1\}$ *that contains zero. Here, the function* $R(z) = 1 + \sum_{n=0}^{s-1} z^{n+1} \boldsymbol{b}\boldsymbol{A}^n \mathbf{1}$ *represents the stability function (see, e.g., [7, Section 238], [1, Section 4.4], [8, Section 5.3]); see also Figure 6.2.*

The left and right pairs of the columns $\tilde{x}(\pi/2)$*,* $\tilde{y}(\pi/2)$ *give the result of the application of one step* $h = \pi/2$ *to systems of differential equations* $\mathrm{d}x/\mathrm{d}t = -y$*,* $\mathrm{d}y/\mathrm{d}t = x$ *and* $\mathrm{d}x/\mathrm{d}t = -y/(x^2+y^2)$*,* $\mathrm{d}y/\mathrm{d}t = x/(x^2+y^2)$*, respectively, with the initial condition* $x(0) = 1$*,* $y(0) = 0$*; see also Figure 6.3. For both systems the exact solution is* $x(t) = \cos t$*,* $y(t) = \sin t$*. For the left columns pair one has* $\tilde{x}(\pi/2) + \mathrm{i}\,\tilde{y}(\pi/2) = R(\mathrm{i}\,\pi/2)$*.*

	s	$10^6 \times T_{11}$	$10^6 \times T_{12}$	$10^6 \times T_{13}$	$\max_{ij}\lvert a_{ij}\rvert$	$\min_j b_j$
C10	18	3.50...	8.14...	13.06...	5.4724...	0.03333...
H10	17	5.27...	17.22...	36.01...	1.0549...	−0.18
O10	17	1.25...	3.01...	4.71...	1.3763...	−0.17892...
F10	17	21.89...	64.01...	113.71...	5.7842...	−0.05
Z10	16	1.42...	21.70...	37.89...	4.9406...	−1.19177...
eq. (5.10)	15	3.49...	8.48...	14.07...	2.2415...	0.03333...

	z_R	$\tilde{x}(\pi/2)$	$\tilde{y}(\pi/2)$	$\tilde{x}(\pi/2)$	$\tilde{y}(\pi/2)$
C10	−3.8269...	−0.00001559...	1.0000226...	0.000093...	1.000561...
H10	−2.7046...	−0.00071183...	1.0004307...	0.011791...	1.007904...
O10	−3.3815...	−0.00006422...	1.0000264...	0.000151...	1.000116...
F10	−2.5279...	−0.00091244...	1.0007372...	−0.004805...	0.996073...
Z10	−4.7240...	−0.00000464...	1.0000090...	−0.004199...	0.997594...
eq. (5.10)	−4.4293...	−0.00000074...	1.0000335...	0.000203...	1.000054...

In H10 there are four non-quadrature node clusters $\{2, 16\}$, $\{3, 15\}$, $\{6, 13\}$, and $\{7, 14\}$; see Figure 6.1, top right panel. They play a role of nested layers of insulation, both from the opening and closing, around the four stages 9, 10, 11, and 12, allowing the latter to have both high stage order (5) and stage co-order (4).

As the Z10 method was found by numerically minimizing $\mathcal{F} = \sum_{\mathrm{t}} \big(\boldsymbol{b}\boldsymbol{\Phi}(\mathrm{t}) - 1/\mathrm{t}!\big)^2$, where the summation goes over all rooted trees t with $|\mathrm{t}| \le 10$ with the value $\mathcal{F} = 0$ being eventually achieved, the method lacks "explainability": The reasons behind its structure are not transparent or easily understandable. The Z10 method contains an idiosyncratic node cluster (S, Q, D) with 5 stages: $S = \{3, 10, 11, 14, 15\}$; see Figure 6.1, bottom left panel. On this cluster it holds that $\boldsymbol{b}|_S \boldsymbol{q}_4|_S = -1.28... \times 10^{-4} \neq 0$ and $\boldsymbol{d}_2|_S \mathbf{1}|_S = 1.62... \times 10^{-5} \neq 0$, and its cluster order and co-order are 4 and 2, respectively. With $\boldsymbol{q}_1|_S = \mathbf{0}$, the vectors $\boldsymbol{q}_2|_S$, $(\boldsymbol{A}\boldsymbol{q}_1)|_S$, $\boldsymbol{q}_3|_S$, and $(\boldsymbol{A}\boldsymbol{q}_2)|_S$ generate a one-dimensional subspace $Q' \subseteq \mathbb{R}^5$. With $\boldsymbol{d}_0 = \mathbf{0}$, the subspace $D' = \mathrm{span}(\boldsymbol{d}_1|_S)$ is also one-dimensional. The subspaces $Q \supseteq Q'$ and $D \supseteq D'$, with $\dim Q + \dim D = 4$, can be chosen in a variety of ways (that is why the cluster type is shown as ${}^{4}_{2}\bullet^{1+}_{1+}$).

The regions of absolute stability of the six methods are illustrated in Figure 6.2. The internal structure of the methods through the progression, the sensitivity to the right-hand side function and the alignment along the trajectory of the intermediate positions $\boldsymbol{X}_i$, $1 \le i \le s$, is illustrated in Figure 6.3.

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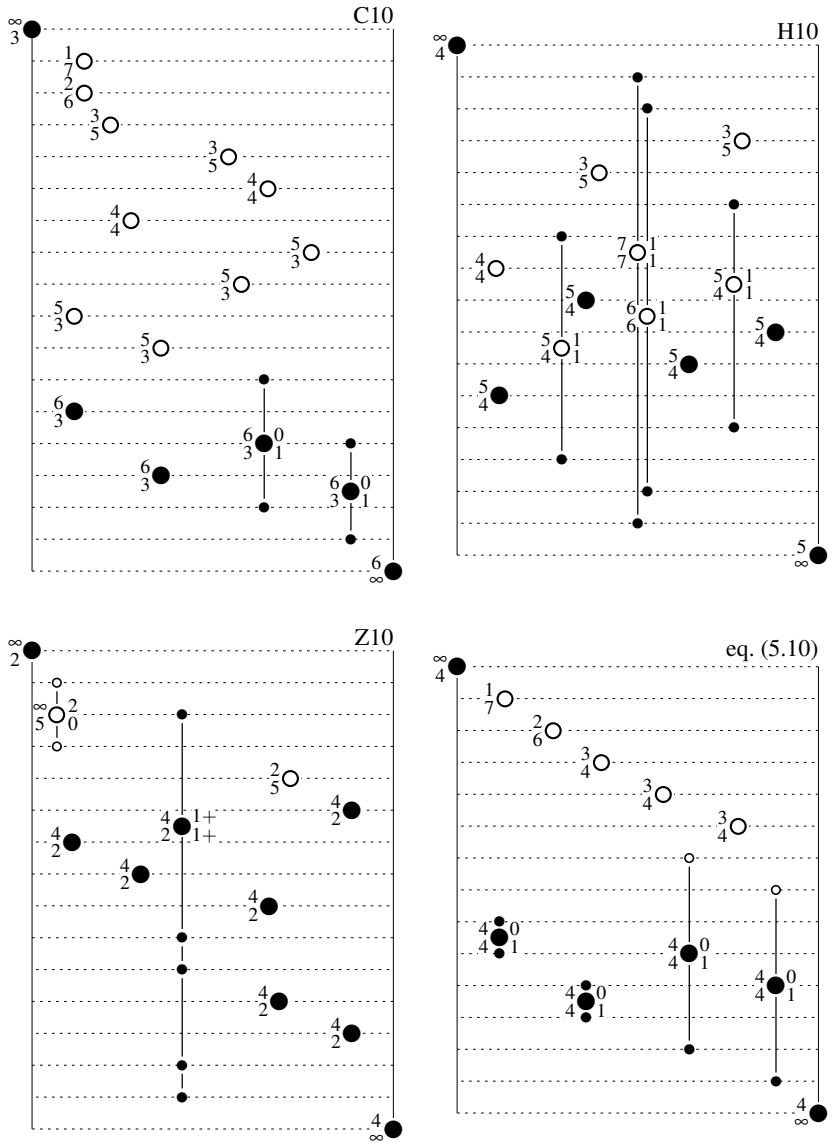

FIG. 6.1. *Node cluster structures of four methods of order* 10*: C10, H10, Z10, and equation (5.10). The dashed lines mark the stages, from* 1 *(upper line) to* s *(lower line). The horizontal axis represents the node positions. Vertical lines connecting small circles correspond to node clusters with more than one stage.*

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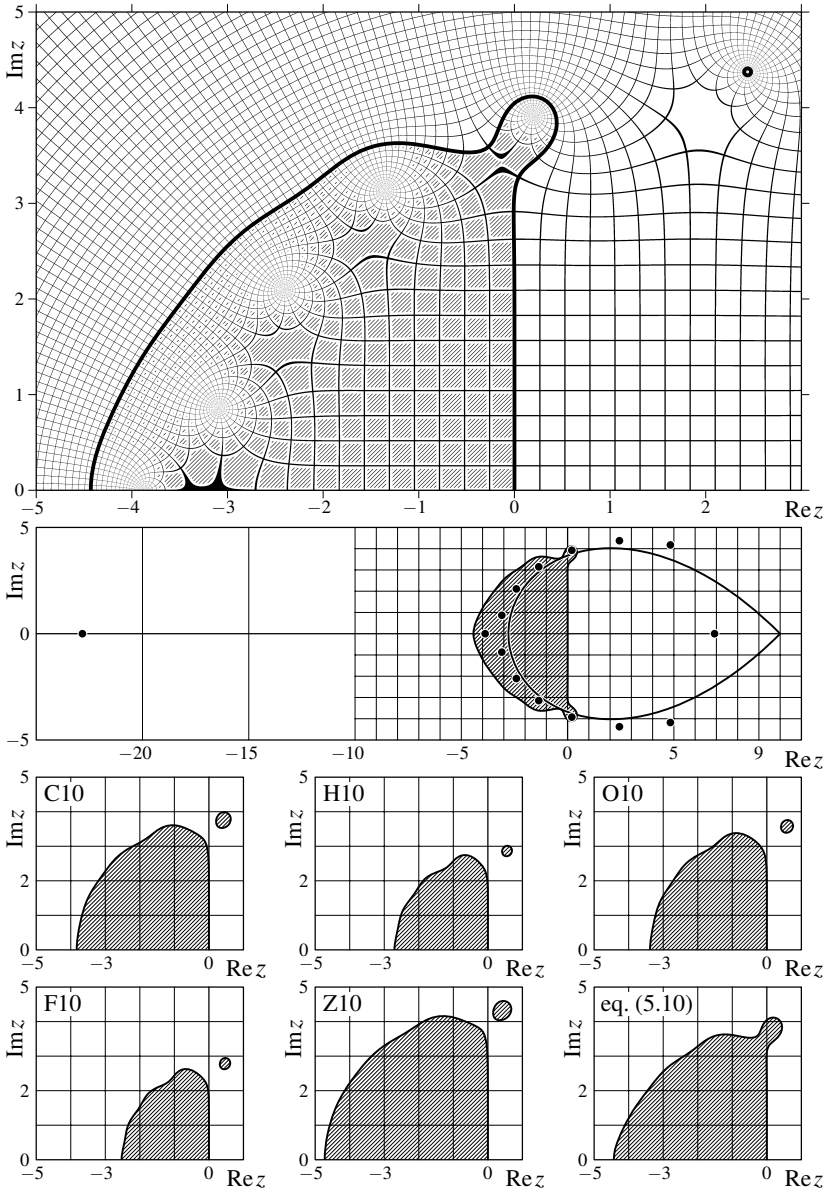

FIG. 6.2. *Regions of absolute stability* $\{z \,|\, |R(z)| \le 1\}$ *of the methods C10, H10, O10, F10, Z10, and that of equation (5.10). The upper and middle panels correspond to the method of equation (5.10). In the upper panel, the thick solid curve marks the boundary of the shaded stability region. The curvilinear grid of lines depicts the regions* $d(\mathrm{Re}\, L(z)) \le 1/40$ *and* $d(\mathrm{Im}\, L(z)) \le 1/40$*, where* $L(z) = (12/\pi)\log R(z)$ *and* $d(x) = \min_{n\in\mathbb{Z}} |x-n| = |x - \mathrm{round}(x)|$ *is the distance to the closest integer. The grid becomes more dense on the left due to the argument principle and numerous zeros of* $R(z)$*. As* $R(z) \approx \exp(z)$ *in the vicinity of* $z = 0$*, the curvilinear grid resembles a square one there. In the middle panel the fifteen points correspond to zeros of the stability function* $R(z)$*, while the solid curve is the Szegő curve* $|z\exp(1-z)| = 1$ *[23] expanded by a factor* 10.

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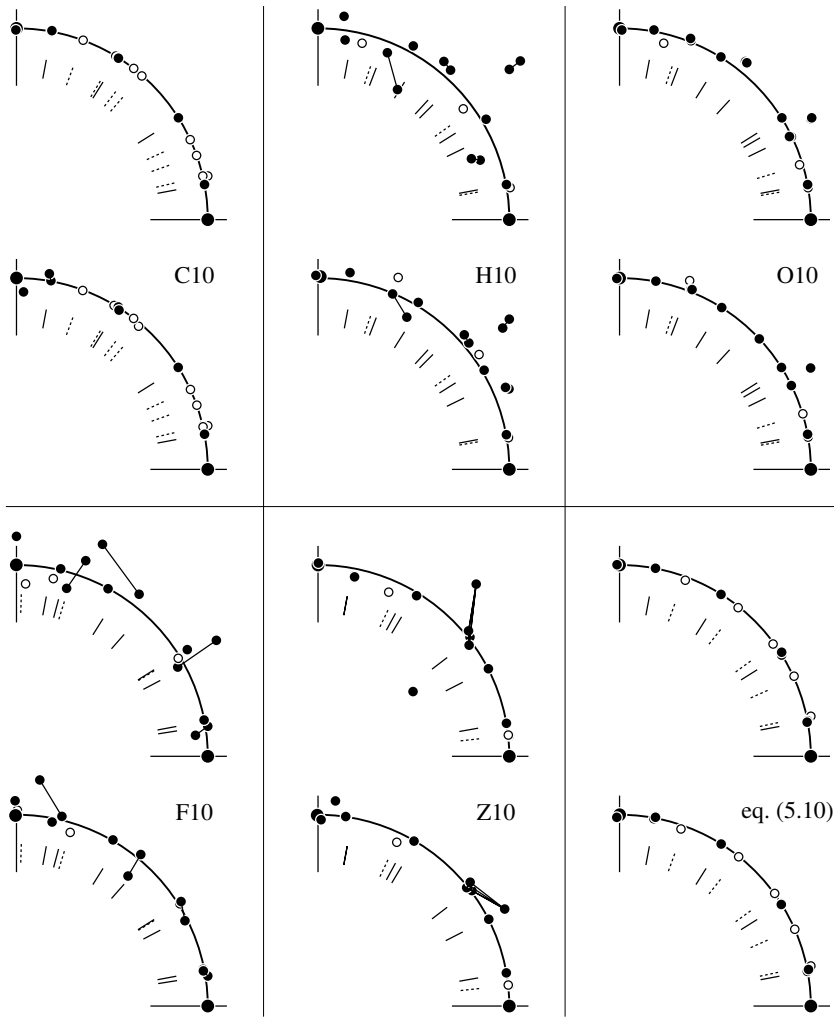

FIG. 6.3. *Application of one step $h = \pi/2$ of the methods C10, H10, O10, F10, Z10, and that of equation (5.10) to systems of differential equations $\mathrm{d}x/\mathrm{d}t = -y$, $\mathrm{d}y/\mathrm{d}t = x$ (upper quarter-circles) and $\mathrm{d}x/\mathrm{d}t = -y/(x^2+y^2)$, $\mathrm{d}y/\mathrm{d}t = x/(x^2+y^2)$ (lower quarter-circles), with the initial condition $x(0) = 1$, $y(0) = 0$. The initial and final points, $(1,0)$ and $\big(\tilde{x}(\pi/2), \tilde{y}(\pi/2)\big)$, respectively, are marked by large black dots. Smaller closed and open dots correspond to the intermediate positions $\boldsymbol{X}_i$, $1 \le i \le s$, with $b_i \neq 0$ and $b_i = 0$, respectively. Within node clusters, these are connected by thin solid lines. Nodes c_i, $1 \le i \le s$, are shown by radial ticks with solid ($b_i \neq 0$) and dashed ($b_i = 0$) lines.*

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